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Topics on Space-Time Topology

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Abstract

The orientability properties of space-times are analysed in detail using elementary algebraic methods. Time, space and charge orientability are discussed and various possible generalisations of charge orientability suggested. There is also a bundle-theoretic analysis of the first two topological properties together with a discussion of spinor-structures from the point of view of the Lorentz bundle of bases over a space-time. A section is devoted to some comments on the topologisation of certain space-times with topologies derived from their causal relations.

In this paper we shall discuss two topics on space-time topology. These are the orientability properties of space-times and the retopologisation of certain space-times with topologies induced by a casual relation. The discussion will be mathematically informal, that is, descriptive rather than axiomatic. The reader is referred to the textbooks of Spanier (1966) and Steen & Seebach (1970) for the definitions and proofs of algebraic and analytic-topological statements.

Apart from the overall orientability or non-orientability of a space-time as a topological manifold, a space-time may or may not possess several other orientability properties. These are topological properties peculiar to space-times: 'future-orientability (Markus, 1955) or 'time-orientability', 'space-orientability' and 'charge orientability' (Geroch, 1969). The first of these properties is geometrical whilst the last is physico-geometrical. We shall discuss possible generalisations of the latter orientability, namely '*I*-orientability' and 'phase-orientability'. Our discussion of the orientability properties will be divided into two parts. In Part 1 we will approach the properties from the point of view of elementary algebraic topology: homotopy, singular homology and singular cohomology. In Part 2 we will discuss the properties from the viewpoint of the more complex and power-

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ful Čech cohomology theory and the Lorentz bundle of bases of a space-time. In this part we shall also briefly discuss group extension theory and the algebraic structure of the Lorentz group.

The importance of the above-mentioned topological properties is that they are about the only topological restrictions that have been proposed as properties that those space-times representing 'reasonable' models of Space-Time should possess.

A space-time is usually defined (Penrose, 1968; Lichnerowicz, 1968) as a smooth, connected, paracompact, Hausdorff 4-manifold X which carries a smooth global Lorentzian tensor field $\mathscr{L}$. A justification of this definition would probably proceed as follows. The 'smooth 4-manifold with smooth Lorentzian tensor field' part of the definition, is the basis of general relativity, piecing together local Minkowski spaces. The topological manifold structure is a rather natural choice for a model of Space-Time (if we make the hypothesis that the universe 'looks the same' everywhere). This is because the group of autohomoeomorphisms of a (connected) topological manifold acts transitively (Spanier, 1966), that is, for each x_0, $x_0' \in X$, there is a homoeomorphism: $(X, x_0) \cong (X, x_0')$, which carries any neighbourhood of x_0 homoeomorphically onto a neighbourhood of x_0'. The connectedness condition (equivalently path-connectedness) is imposed because one would like to think that Space-Time consists of 'one piece' or that any two space-time points could be connected by a curve $\gamma: x_0 \rightsquigarrow x_0'$. Lastly, the conditions of paracompactness and T_2 separability are imposed mainly for mathematical convenience: any paracompact Hausdorff manifold admits a global Riemannian metric tensor field and a countable atlas.

Part 1. Orientability Properties

1.1. *Time-Orientability*

In the following, we shall always assume that a space-time X has (at least) the structures listed above. The tangent bundle (Brickell & Clarke, 1970) of X' $\xi_T(X)'$, will be denoted by

$$\xi_T(X) \equiv (T(X), X, T(X, x_0), \pi_T)$$

Here $T(X)$ is the totality of vectors tangent to X, π_T is the projection of $T(X)$ onto X which assigns a base point in X to a tangent vector over X. The fibres $\pi_T^{-1}\{x_0\} = T(X, x_0)$, are the linear tangent spaces to X at a given point $x_0 \in X$. The fibre $T(X, x_0)$ is linearly diffeomorphic to $\mathbb{R}^4$, which we shall usually identify with Minkowski space. The Lorentzian tensor field $\mathscr{L}$ allows us to categorise the vectors over X as time-like, space-like or light-like according to

$$\mathscr{L}_{\pi_T(v)}(v, v) > 0, < 0, \text{ or } = 0$$

A vector field on X is a global smooth section of the bundle projection π_T; that is, a smooth function $V: X \to T(X)$ such that $\pi_T \circ V = \mathbb{1}_X$. Thus a vector field is a global smooth choice of a vector $V_x \in T(X,x)$ for $x \in X$. A vector field will be called 'time-like' iff the function $\mathscr{L}(V,V)$: $X \to \mathbb{R}$; $x \mapsto \mathscr{L}_x(V_x, V_x)$ is positive definite. The Lorentzian tensor allows one to define a bundle $\xi_t(X)$ of time-like vectors over a space-time X

$$\xi_t(X) \equiv (t(X), X, t(X,x_0), \pi_t)$$

The total space $t(X)$ of $\xi_t(X)$ is the open submanifold of time-like vectors over X: $\mathscr{L}^{*-1}(]0\rangle)$, where $\mathscr{L}^*$ is the function from $T(X)$ to $\mathbb{R}$ which sends $v \in T(X)$ to $\mathscr{L}_{\pi_T(v)}(v,v)$. The projection π_t is the projection π_T restriction to $t(X)$. Its fibres $t(X,x_0)$ are the sets of time-like vectors over $x_0 \in X$.

The interesting point about this bundle is that its fibres are disconnected. Let $v_\pm(x_0)$ be the two components of the fibre over x_0. A choice of component $v_\pm(x_0)$ over x_0 represents a selection of a 'time-sense' at x_0. If there is a section of $\xi_t(X)$, a global choice of time-like vector over X, we may make a continuous choice of component $v_\pm(x)$ over X; that is, a global sense of time

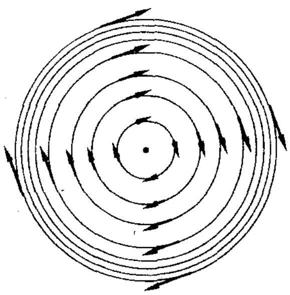

Figure 1

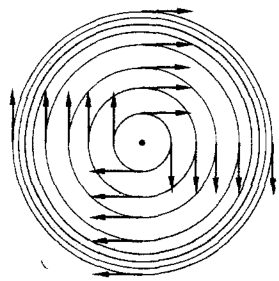

Figure 2

over the whole space-time manifold. Not all space-times have such a convenient property. Clearly, if we were able to make the above construction X would have to carry a global non-vanishing vector field. Not all smooth manifolds can carry such a vector field. The 'Hairy Ball Theorem' (Maunder, 1970) tells us, for example, that the two-sphere S^2 does not. To see this intuitively, attach a unit tangent vector to each point S^2 imbedded in $\mathbb{R}^3$ around lines of latitude S^1. When we reach a pole, the vectors will have 'nowhere to point' (Fig. 1). To make a system of tangent vectors point in smoothly prescribed directions, their lengths have to vanish at the poles (Fig. 2). One can show that any smooth vector field on S^2 has to vanish at least once (Eilenberg, 1963).

One may also show that no even-dimensional sphere can carry a non-vanishing vector field. Hence, in particular S^4 cannot. Any model of the universe which is topologically S^4 is not 'future orientable', in the sense that we cannot construct a non-vanishing (time-like) vector field.

If we define $v(X)$ as the set of all components $v_\pm(x)$ and give it the quotient topology of $t(X)$ under the relation $v \sim v'$ iff v and v' lie in the same fibre over X and in the same component of the fibre, the local existence of sections of $\xi_t(X)$, time-like vector fields defined along certain open sets of X, ensures that $v(X)$, along with the obvious projection, is a two-fold covering space (Spanier, 1966) of X. Let us agree to call a space-time 'time-orientable' iff there is a section of the latter covering projection. Then in a time-orientable space-time one may make a global continuous choice of time-sense. Equivalently (Spanier, 1966) each homotopy class of loops $|\gamma| \in \pi_1(X, x_0)$, where $\pi_1(X, x_0)$ is the first homotopy group of X at x_0 (Spanier, 1966), should induce the identity auto-morphism of the fibre $v(X, x_0) = \{v_\pm(X_0)\}$ over

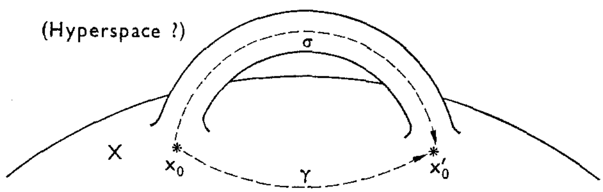

Figure 3

$x_0 \in X$. Again, this condition is equivalent to the requirement that any homotopy class paths $|\gamma|$ for $\gamma: x_0 \rightsquigarrow x_0'$ induces the same homoeomorphism

$$|\gamma|^{\#}: v(X, x_0) \cong v(X, x_0')$$

or that the structure group (Spanier, 1966) of the principal fibre bundle (the time-sense bundle)

$$\tau(X) = (v(X), X, v(X, x_0), \mathbb{Z}_2, p)$$

should be reducible to the identity. This latter gives us the following graphic interpretation. Given space-time points x_0, $x_0' \in X$, any path $\gamma: x_0 \rightsquigarrow x_0'$ allows one to carry a choice of time-sense from x_0 to x_0' along γ: $v_0 \mapsto |\gamma|^{\#}(v_0)$ for $v_0 \in v(X, x_0)$ and, as the notation suggests, the time-sense only depends on the homotopy class $|\gamma|$ of the path γ. Suppose σ is a path from x_0 to x_0' non-homotopic to γ. Carrying the time-sense $v_0 \in v(X, x_0)$ along σ yields a time-sense $|\sigma|^{\#}(v_0) \in v(X, x_0')$. Physically, one would like X to be such that $|\sigma|^{\#}(v_0) = |\gamma|^{\#}(v_0)$ or that $|\gamma|^{\#}(v_0)$ should be independent of γ. That is, X should be time-orientable. Otherwise, if an interstellar voyager sets out from x_0 to x_0' along γ and another one along σ (Fig. 3) then if $|\gamma|^{\#}(v_0) \neq |\sigma|^{\#}(v_0)$, according to the former voyager, the latter would be 'getting younger', and vice versa!

We have already seen that if a space-time carries a time-like vector field V, it is time-orientable, for $x \mapsto$ 'component of V_x' is a section of the time-sense bundle. Conversely, a time-orientable space-time carries a time-like vector field. If X is a space-time, we shall denote its three-sphere bundle $\xi_\sigma(X)$ of unit tangent vectors by

$$\xi_\sigma(X) \equiv (\sigma(X), X, \sigma(X, x_0), \pi_\sigma)$$

This bundle has total space $\sigma(X)$ of all unit tangent vectors with respect to a Riemannian metric and the three-sphere fibres $\sigma(X, x_0)$ are the unit vectors over $x_0 \in X$. If we express the Lorentz tensor $\mathscr{L}$ in its diagonal form over x_0, a three-dimensional picture of the tangent (Minkowski) space (Fig. 4) convinces us (Geroch, 1969) that in any space-time there is a vector field-up-to-a-sign θ which assigns to $x_0 \in X$ the up-to-a-sign vector in $T(X, x_0)$ with unit (Riemannian) and (Lorentzian) 'norms'. If X is time-orientable, one may make a continuous choice of future cone and hence of a sign of θ. The composed function is then a unit time-like vector field on X. Thus a connected paracompact Hausdorff space-time is time-orientable iff it carries a unit time-like vector field.

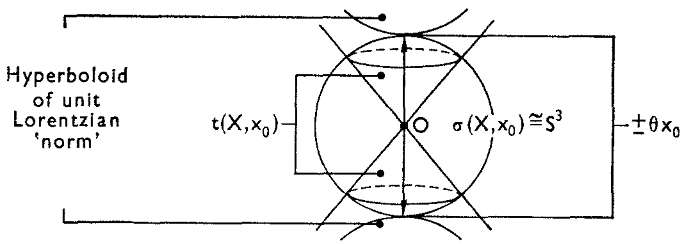

Figure 4

There is a generalisation of the Hairy Ball Theorem to the effect that a compact (Hausdorff) manifold carries a unit vector field iff its Euler–Poincaré characteristic (Spanier, 1966) $\chi(X)$ vanishes. Thus space-times topologically $\mathbb{R}P^4$, S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, ... cannot be time-orientable, since they have characteristics of 1, 2, 3 and 4 respectively. If one starts with a given Hausdorff 4-manifold X one may or may not be able to give X the structure of a time-orientable space-time. One can, however, always construct the structure of a time-orientable space-time on a manifold carrying a non-vanishing vector field. If R denotes a Riemannian metric tensor on such a space-time, and V the non-vanishing vector field, one may define a Lorentzian tensor field $\mathscr{L}$ by

$$\mathscr{L}: x \mapsto R_x - R_x^*(V_x) \otimes R_x^*(V_x)/2R_x(V_x, V_x)$$

[here R_x^* are the local isomorphisms of the tangent spaces $T(X, x)$ onto the cotangent spaces $T^*(X, x)$ induced by R]. With respect to $\mathscr{L}$, V is a nowhere-vanishing time-like vector field.

Given any space-time X, there is a group-homomorphism: $\varphi_X \in \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2)$ defined as follows. If $|\gamma|$ is the homotopy class of a loop γ in X at x_0, $|\gamma|$ induces an autohomoeomorphism $|\gamma|^\#$ of the fibre $v(X, x_0)$ in the time-sense bundle $\tau(X)$. $|\gamma|^\#$ is defined by the rule

$$|\gamma|^\# : v_0 \mapsto \gamma^*_{v_0} \qquad \text{for } v_0 \in v(X, x_0)$$

Here, $\gamma^*_{v_0}$ is the unique lift (Spanier, 1966) of γ from the time-sense v_0. It is trivial to verify that the function

$$\varphi_X : |\gamma| \mapsto |\gamma|^{\#} \in \mathrm{Sym}(v(X, x_0)) \cong \mathbb{Z}_2$$

is a homomorphism. Also, it is clear that X is time-orientable iff $\varphi_X = 1$ the trivial homomorphism of $\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2)$. For if $\varphi_X = 1$, any loop in X at x_0 lifts to a loop in $v(X)$, in which case a section of the covering projection exists (Spanier, 1966). If such a section s exists and γ is a loop in X at x_0, $\gamma^*_{s(x_0)}$ is a loop in $V(X)$ at $s(x_0)$ by the unique path-lifting property (Spanier, 1966), and hence $\varphi_X = 1$.

Of course, we are guaranteed that X is time-orientable if $\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2) = 1$. Thus if X is topologically such that $\pi_1(X, x_0)$ has no invariant subgroup of index 2, X must be time-orientable. For given $\pi_1(X, x_0)$ has no invariant subgroup of index 2, $\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2) = 1$. To see this, suppose that $\varphi \in \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2)$ and $\varphi \neq 1$, then φ is an epimorphism and $\mathrm{Ker}(\varphi) \lhd \pi_1(X, x_0)$ is an invariant subgroup of index 2.

This result may be translated into the language of (singular) cohomology theory as follows. Firstly, we note (Scott, 1964) that there is an isomorphism of abelian groups:

$$\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2) \cong \mathrm{Hom}(\mathrm{ab}(\pi_1(X, x_0)), \mathbb{Z}_2)$$

where $\mathrm{ab}(\pi_1(X, x_0))$ is the 'abelianisation' of the group $\pi_1(X, x_0)$: $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$. But (Greenberg, 1967) from

$$\mathrm{ab}(\pi_1(X, x_0)) \cong H_1(X, \mathbb{Z})$$

where $H_1(X, \mathbb{Z})$ is the first integral singular homology group we get

$$\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}_2)$$

From the short-exact sequence of the universal coefficient theorem for cohomology (Spanier, 1966)

$$\mathrm{Ext}(H_0(X, \mathbb{Z}), \mathbb{Z}_2) \rightarrowtail H^1(X, \mathbb{Z}_2) \twoheadrightarrow \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}_2)$$

and the fact that $H_0(X, \mathbb{Z}) \cong \mathbb{Z}$ because X is path connected, we obtain

$$\mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}_2) \cong H^1(X, \mathbb{Z}_2)$$

where the latter is the first singular mod-2 cohomology group of X. If the image of φ_X under the above isomorphism is Ψ_X, then Ψ_X alternatively represents the obstruction to the time-orientability of a given space-time X: X is time-orientable iff $\Psi_X = 0$.

The latter considerations motivate the following definition. A space-time X is called T-orientable iff the O-sphere bundle $\tau(X)$ is orientable in the sense of Spanier (1966). Equivalently (Spanier, 1966), X is T-orientable iff the homoeomorphism $|\gamma|^{\#}$ induced by any loop class $|\gamma| \in \pi_1(X, x)$ induces the identity automorphism of the reduced (Spanier, 1966) cohomology groups ${}_R H^0(v(X, x), \mathbb{Z})$ of the fibre O-spheres $v(X, x)$. Recall that since

$v(X,x_0) \cong S^0$, ${}_RH^0(v(X,x_0),\mathbb{Z}) \cong \mathbb{Z}$, being generated by the duals $|v_0{}^*|$, say, of the singular O-simplexes $v_0{:}\Delta^0 \mapsto v_0$. Carrying such a generator around a loop in X should result in returning with the same generator: ${}_RH^0(|\gamma|^\#) \times (|v_0{}^*|) = |v_0{}^*|$ (instead of $-|v_0{}^*|$). One may extend this definition to calling a space-time T-orientable over a ring R iff $\tau(X)$ is an orientable O-sphere bundle over R (Spanier, 1966). Using the equivalent definition that X is T-orientable over R iff any $|\gamma| \in \pi_1(X,x_0)$ induces the identity automorphisms of ${}_RH^0(v(X,x_0),R)$, it follows from the fact that $\mathbb{Z}_2$ has only one generator, that any space-time is T-orientable over the field $\mathbb{Z}_2$. There is a necessary and sufficient condition that a space-time X be T-orientable (over $\mathbb{Z}$). This is, that the first Stiefel–Whitney class $w_1(\tau(X))$ of $\tau(X)$, $w_1(\tau(X)) \in H^1(X,\mathbb{Z}_2)$ should vanish. Using this condition one may show that T-orientability over $\mathbb{Z}$ and time-orientability coincide. If X is time-orientable, there is a section of $\tau(X)$, but in this case, one may show that the fundamental class (Spanier, 1966) $\Omega_{\tau(X)}$ of $\tau(X)$ vanishes. But for O-sphere bundles, the fundamental class coincides with the first Stiefel–Whitney class. Thus a time-orientable space-time is T-orientable over $\mathbb{Z}$. Conversely, if X is T-orientable over $\mathbb{Z}$, it is time-orientable. For then the homoeomorphisms $|\gamma|^\#$ induce the trivial automorphisms of ${}_RH^0(v(X,x_0),\mathbb{Z})$; $|\gamma| \in \pi_1(X,x_0)$. That is, ${}_RH^0(|\gamma|^\#) \times (|v_0{}^*|) = |v_0{}^*|$. Using the Kronecker product, we get

$$H^0(|\gamma|^\#)(|v_0{}^*|)(|v_0|) = |v_0{}^*|(|v_0|) = 1 = |v_0{}^*|(|\gamma|^\#(|v_0|))$$

therefore, $|v_0{}^*|(|\gamma|^\#(v_0)) = 1 \Rightarrow |\gamma|^\#(v_0) = v_0$, or any loop in X lifts to a loop in $\tau(X)$. Consequently, X is time-orientable iff $w_1(\tau(X)) = 0$.

1.2. *Space-Orientability*

The concept of the space-orientability of a space-time is dual to the concept of time-orientability. It is defined as follows. We have seen that any space-time X carries a line-bundle. That is, there is a one-dimensional linear subspace smoothly assignable to each tangent Minkowski space over X. These are the linear spaces generated by the unit vectors $\pm\theta_x$: $\mathbb{R}_*(\theta_x) = \mathbb{R}_*(-\theta_x)$. We may form a quotient bundle of the tangent bundle $\xi_T(X)$ as follows. Define $v \div v'$ for v, $v' \in T(X)$ iff $\pi_T(v) = \pi_T(v')$ and $v - v' \in \mathbb{R}_*(\theta\pi_T(v))$. This bundle has fibres $T(X,x_0)/\mathbb{R}_*(\theta_x)$ which are three-dimensional linear subspaces of $T(X,x_0)$. [Note that out of any basis of $T(X,x_0)$ including either of $\pm\theta_x$, we may choose the other three vectors as space-like.] We denote $T(X,x_0)/\mathbb{R}_*(\theta_x)$ by $S(X,x_0)$. It is clear that we may place a quotient metric on the new quotient bundle, and hence we may form a two-sphere bundle of unit vectors. The fibres of the latter will be called 'celestial spheres'. S^2 is an orientable 2-manifold (Spanier, 1966; Greenberg, 1967). If we denote the unit vectors of $S(X,x_0)$ by $s(X,x_0)$, the two spheres $s(X,x_0)$ have orientations $\pm\sigma_x$, which we shall shortly define more precisely. Because one may locally define consistent orientations of these celestial spheres, one can construct a two-fold covering space of X consisting of the totality of the $\pm\sigma_x$ with a suitable topology. X is called 'space-orientable iff there is a section of this covering space.' Clearly, the analysis of space-orientability is analogous to that of time-orientability. One may define a homomorphism $\bar{\varphi}_X \in \mathrm{Hom}(\pi_1(X,x_0),\mathbb{Z}_2)$ or a cohomology class $\bar{\Psi}_X \in H^1(X,\mathbb{Z}_2)$ representing the obstruction to a given space-time being space-orientable. 'S-orientability' over a ring R is also suggested, a space-time being called S-orientable over R iff the bundle of orientations of celestial spheres is an R-orientable O-sphere bundle.

An orientation of an n-manifold M is a choice of generator of the homology groups $H_n(M,M\backslash\{m\};\mathbb{Z})$ for $m \in M$ defined continuously over M. S^2 is an orientable 2-manifold because there is a class $U \in H_2(S^2;\mathbb{Z})$ such that if $s \in S^2$ and J_s: $(S^2,\varphi) \subset (S^2,S^2\backslash\{s\})$ the homology class $H_2(J_s)(U) \in H_2(S^2,S^2\backslash\{s\})$ is a generator. U is called an orientation of S^2. A connected orientable manifold has just two orientations. $\pm\sigma_x \in H_2(s(X,x),\mathbb{Z})$ are the orientations of the celestial spheres $s(X,x)$. A space-time is thus called space-orientable iff carrying a generator σ_0 of $H_2(s(X,x_0),\mathbb{Z}_2)$ around a loop in X results in returning with the same generator.

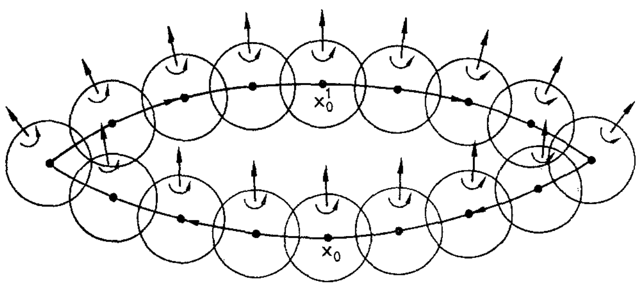

Figure 5

Alternatively, the transport of senses of left and right around the space-time should be independent of path chosen. In a non-space-orientable space-time, if two interstellar voyagers set out from x_0 to x_0' along non-homotopic paths, they could be presented with the mirror images of each other on reaching x_0'!

The definition of the orientability of the celestial spheres is closely allied to the following construction. Let us agree to call a space-time X 'S^*-orientable' iff the two-sphere bundle of celestial spheres constructed above is an orientable two-sphere bundle (Spanier, 1966). Now, any path γ in X induces (Spanier, 1966) homotopy equivalences $|\gamma|^{\#}: s(X,x_0) \simeq s(X,x_0')$ for $\gamma: x_0 \rightsquigarrow x_0'$. Equivalently, X is S-orientable iff any loop γ in X induces the identity automorphism of the (reduced) cohomology group

$$_RH^2(s(X,x),\mathbb{Z}) = H^2(s(X,x),\mathbb{Z})$$

Because $H^2(s(X,x),\mathbb{Z}) \cong H_2(s(X,x),\mathbb{Z})$, the implication is that carrying any orientation of $s(X,x)$ around a loop in X results in returning with the same orientation. One may show (Spanier, 1966) that X is S^*-orientable iff the first Stiefel–Whitney class of the bundle of celestial spheres vanishes: $w_1(\xi_s(X)) \in H^1(X,\mathbb{Z}_2)$. It is clear that X is S^*-orientable iff X is space-orientable.

We shall return to time orientability and space-orientability in the second part. Next, we turn to various physico-geometric orientability properties.

1.3. *Physico-Geometric Orientabilities*

The prototype of the orientabilities we shall be considering in this section is charge-orientability as formulated by Geroch (1969). Over each space-time point x there lies a pair $\pm e_x$ of charges. One makes the very reasonable hypothesis that there are neighbourhoods of any space-time points such that along these open sets one may make a (continuous) choice of either of $\pm e$. In this way, with a suitable topology the totality of all charges over the points of X is a two-fold covering space of X: the charge bundle on which $\mathbb{Z}_2$ acts via the charge-conjugation operator C. X is called 'charge orientable' iff there is a section of the covering projections, or equivalently if loops in X lift to loops in the charge-covering space, or if paths in X induce the same homoeomorphism of the fibres over the end points. The analysis of charge-orientability is clearly closely analogous to those of time and space-orientability, the obstruction to X being charge-orientable being represented by a homoeomorphism $\tilde{\varphi}_X \in \mathrm{Hom}(\pi_1(X,x_0),\mathbb{Z}_2)$ or by a mod-2 cohomology class $\tilde{\Psi}_X \in H^1(X,\mathbb{Z}_2)$. Clearly, if X is such that $\pi_1(X,x_0)$ has no invariant subgroup of index 2 or if $H^1(X,\mathbb{Z}_2)=0$, X is time-orientable space-orientable and charge-orientable. Even more, X is an orientable manifold, the orientability of a space-time manifold being analysed in one approach (Greenberg, 1967) by constructing a two-fold orientation covering space of all generators of the groups $H_4(X,X\backslash\{x\})$ for $x \in X$. By excision, $H_4(X,X\backslash\{x\}) \cong H_4(B,B\backslash\{x\}) \cong H_3(B\backslash\{x\}) \cong H_3(\partial B')$ where $B \supset B'$ is coordinate neighbourhood around x with $Cl(B') \subset B$ and $\partial B' \cong S^3$; generators of the former homology group carrying through the chain of isomorphisms to a generator of $H_3(\partial B')$ representing an orientation of the boundary three-sphere of the closed coordinate ball B' around x. The obstruction to X being orientable is representable by a $\varphi_X' \in \mathrm{Hom}(\pi_1(X,x_0),\mathbb{Z}_2)$ or a $\Psi_X' \in H^1(X,\mathbb{Z}_2)$. If X is compact, Ψ_X' is the first Stiefel–Whitney class of X (Spanier, 1966).

It is not a large extrapolation from the definition of charge orientability to discuss slightly more complicated analogous orientabilities. For instance, using local sections one may construct a three-fold covering space of X consisting of the totality of I-spin-1 multiplets (pions) over the points of X, X being called I-orientable iff the covering space admits a section in which case we can globally and continuously distinguish, say, π^0 from $\pi^\pm$. To be able to globally and continuously distinguish all three, the structure group $\pi_1(X,x_0)$ of the I-bundle should be reducible to the identity. Similarly, for higher $SU(2)$ multiplets and $SU(3)$ multiplets one may construct covering spaces where fibres are the local multiplets and investigate the possibility of globally being able to split the multiplets using our local observation of the existence of sections.

The following considerations arise from quantum mechanics. The quantum-mechanical state of a system is represented by a vector ray ψ_x over $x \in X$. ψ_x is topologically S^1. Given that we may make a local choice of a vector $\phi_x \in \psi_x$, under what conditions on X can one make a global choice of phase or selection of ϕ_x? If we use the local choices to construct a one-sphere bundle (Spanier, 1965) or 'phase bundle' over X does the bundle necessarily admit a section? One can show that a necessary condition is that the fundamental class (Spanier, 1966) of the phase bundle associated with ψ vanishes. In this case the fundamental class is a cohomology class of the group $H^2(X, \mathbb{Z}_2)$.

Part 2. The Bundle Theoretic Approach

In this part of the paper we consider space and time-orientability from the viewpoint of the Lorentz bundle of bases over a space-time. We also discuss the concept of 'spinor-structures' (Geroch, 1968). Roughly, a spinor-structure on a space-time is a two-fold covering space of the Lorentz bundle $\xi_{L+\uparrow}(X)$ of a space and time orientable space-time with structure group $SL(2, \mathbb{C})$ such that the restriction of the covering map to any fibre $SL(2, \mathbb{C})_x$ for $x \in X$ is the universal covering map $SL(2, \mathbb{C})_x \to L + \uparrow_x$, the structure group of the Lorentz bundle. One can justify the existence of such a structure over a space-time using arguments based on the Aharanov–Susskind 'Gedanken' experiment (Aharanov & Susskind, 1967; see also Geroch, 1968 and Penrose, 1968). Such arguments are ultimately based on the spinorial nature of spin-$\frac{1}{2}$ elementary particles such as the electrons and the neutrino.

To prepare the ground for the discussion, we present below a rapid survey of some of the mathematical tools we shall be using. First, there is a discussion of presheaves of modules and groups on a topological space and the elements of the Čech cohomology theory, including the Čech analogue of the 'Bockstein' exact coefficient sequence of singular cohomology. Then, principal fibre bundles are defined, along with their reduction, extension and the use of Čech cohomology in their classification. Next, there is a brief discussion of group extensions with a non-abelian kernel and the algebraic structure of the Lorentz group L. These results are then applied to the study of the reducability and extendability of the Lorentz bundle and the connection of these operations with the orientability properties just mentioned.

2.1. *Presheaves and Čech cohomology*

(a) A presheaf (Spanier, 1966; Eilenberg, 1963) is a contrafunctor S from the category $\tau(X)$ of open sets and inclusion mappings of a topological space X to a category C which possesses a zero object O such that $S(\phi) = 0$. Thus for each open set $V \in \tau(X)$ there is an object $S(V) \in C$ and if $i_V^{V'}$: $V \subset V'$ is an inclusion mapping of $\tau(X)$, $S(i_V^{V'}): S(V') \to S(V)$ is a homomorphism of C such that if $V \subset V' \subset V''$, $S(i_V^{V''}) = S(i_{V'}^{V''}) \circ S(i_V^{V'})$ and $S(i_V{}^V) = \mathbb{1}_{S(V)}$. The homomorphisms $S(i_V^{V'})$ for $i_V^{V'}: V \subsetneq V'$ are called 'restriction maps'. For example, one may take S to be a constant presheaf. Such a presheaf assigns a constant object to C to each open set V of X, $S(V) = G$, say. One denotes such a presheaf by $\mathbf{G}$.

A homomorphism between presheaves S, S' over X is the giving of homomorphisms $H_V: S(V) \to S'(V)$ in C such that whenever $i_V^{V'}: V \subsetneq V'$, the following diagram commutes

$$\begin{array}{ccccc} & S(V') & \xrightarrow{S(i_V^{V'})} & S(V) & \\ H_{V'} & \downarrow & & \downarrow & H_V \\ & S'(V') & \xrightarrow[S'(i_V^{V'})]{} & S'(V) & \end{array}$$

That is, $H: S \to S'$ is a natural transformation of functors. For example, if S and S' are constant presheaves $\mathbf{G}$, $\mathbf{G}'$ over X, a homomorphism $\boldsymbol{\theta}: \mathbf{G} \to \mathbf{G}'$ is induced by any homomorphism $\theta: G \to G'$ in C.

Suppose that C is a 'nice' category such as the category of R-modules in a ring R or the category of groups. Given presheaves S, S' on X with values in C one may define new presheaves $\mathrm{Ker}(\theta)$ and $\mathrm{Im}(\theta)$ for any homomorphism $\theta: S \to S'$ by setting $\mathrm{Ker}(\theta)_V \equiv \mathrm{Ker}(\theta_V)$ for $\theta_V: S(V) \to S'(V)$. It is straightforward to verify that $\mathrm{Ker}(\theta)$ and $\mathrm{Im}(\theta)$ thus defined are in fact presheaves on X. Hence, one may talk of exact sequences of presheaves over a topological space, for example, short-exact sequences: $S \rightarrowtail^{\alpha} S' \xrightarrow{\beta}\!\!\!\!\rightarrow S''$. Clearly, an exact sequence of C induces an exact sequence of constant presheaves over X: $G \rightarrowtail^{\beta} G' \xrightarrow{\alpha}\!\!\!\!\rightarrow G'' \Rightarrow \mathbf{G} \rightarrowtail^{\alpha} \mathbf{G}' \xrightarrow{\beta}\!\!\!\!\rightarrow \mathbf{G}''$. Alternatively, there is a functor imbedding C in the category of presheaves over X.

(b) Let $\mathrm{Cov}(X)$ denote the category of open covers of a topological space X. The objects of $\mathrm{Cov}(X)$ are the open covers of X and via the relation of refinement: $v < v' \in \mathrm{Cov}(X)$ iff $\forall\ V \in v\ \exists\ V' \in v'$ and $V \subset V'$ (v' refines 'v'), the morphisms F: $v \to v'$ iff $v < v'$ are defined by $V \subset F(V)$. In fact, $\mathrm{Cov}(X)$ is directed (Greenberg, 1967) under the relation of refinement. Given a presheaf S from X to C (a nice category), the Čech cohomology of X in S is defined as follows. If $v \in \mathrm{Cov}(X)$, define a cochain complex $C(v, S)$ by defining the 'module' of degree q to be the module of all functions ϕ which assign to a $(q+1)$-tuple: $V_0, \ldots, V_q$ of opens of v with

$$\bigcap_{i=0}^{q} \{V_i\} \neq \varnothing$$

an element $\phi(V_0, \ldots, V_q)$ of

$$S\left(\bigcap_{i=0}^{q} \{V_i\}\right)$$

Addition is defined pointwise and the coboundary homomorphisms from degree q to degree $q+1$ are

$$\delta^q(\phi):(V_0,\ldots,V_{q+1}) \mapsto \sum_{i=0}^{q+1}(-1^i)\,\phi(V_0,\ldots,\hat{V}_i,\ldots,V_{q+1})|\bigcap_{i=0}^{q+1} V_i$$

where $|$ denotes restriction. It is easily verified that $\delta^q \circ \delta^{q-1} = 0$. The modules $H^q(v,S)$ are defined to be $\mathrm{Ker}(\delta^q)/\mathrm{Im}(\delta^{q-1})$. If $v < v'$ there is a homomorphism $r^v_{v'}$: $H^q(v,S) \to H^q(v',S)$ which behaves properly under refinement in such a way that $\langle H^q(v,S)\rangle v \in \mathrm{Cov}(X)$ is a direct system (Spanier, 1966; Greenberg, 1967) over $\mathrm{Cov}(X)$. The Čech cohomology groups of X in S are defined as the direct limit: $\lim \to \langle H^q(v,S)\rangle \equiv \check{H}^q(X,S)$, the direct-limit of the groups $H^q(v,S)$.

If S is a constant presheaf of non-abelian agroups on X, although the higher cohomology groups are not defined in general; the Čech groups $\check{H}^0(X,\mathbf{G})$ and $\check{H}^1(X,\mathbf{G})$ are always defined.

Any homomorphism h: $S \to S'$ of presheaves over X induces a homomorphism h^*: $\check{H}^q(X,S) \to \check{H}^q(X,S')$ in a functorial way. We shall need the following Čech analogue of the Bockstein exact coefficient sequence of singular cohomology (Spanier, 1966) associated with an exact sequence $S \overset{\alpha}{\rightarrowtail} S' \overset{\beta}{\twoheadrightarrow} S''$ of presheaves over X:

$$\cdots \xrightarrow{\beta_*} \check{H}^{q-1}(X,S'') \xrightarrow{\delta_*^{q-1}} \check{H}^q(X,S) \xrightarrow{\alpha_*} \check{H}^q(X,S') \xrightarrow{\beta_*} \check{H}^q(X,S'') \xrightarrow{\delta_*{}^q} \check{H}^{q+1}(X,S) \xrightarrow{\alpha_*} \cdots$$

The homomorphisms $\delta_*{}^q$: $\check{H}^q(X,S'') \to \check{H}^{q+1}(X,S)$ are called the connecting homomorphisms of the exact sequence of presheaves. (Because they connect the three term exact sequences built on the former.)

2.2. *Classification of Principal Fibre Bundles*

(a) A principal fibre bundle ξ over a topological space X is a collection $\xi = (E, X, G, \pi)$. Here, E (the total space of ξ) is a topological space on which the topological group G (the structure group of ξ) acts effectively in such a way that $\pi: E \to X = E/G$ and for each point of X there is a neighbourhood. V

and a homoeomorphism $h_V: \pi^{-1}(V) \cong V \times G$ with $p_V \circ h_V = \pi_V \equiv \pi|\pi^{-1}(V)$; p_V being the projection of $V \times G$ onto V. Let $\lambda: E \times G \to E$ be the continuous action of G on E.

A homomorphism of principal fibre bundles $\xi \to \xi'$ over X is a pair (f, φ), where f is a continuous function $E \to E'$ and φ a continuous homomorphism $\varphi: G \to G'$ such that the following diagram commutes:

$$\begin{array}{ccccc} & E \times G & \xrightarrow{\lambda} & E & \\ f \times \varphi & \downarrow & & f \downarrow & \searrow^{\pi} \quad X \\ & E' \times G' & \xrightarrow[\lambda']{} & E' & \nearrow_{\pi'} \end{array}$$

A homomorphism of principal fibre bundles is called a reduction of the structure group iff f is an inclusion map and φ an inclusion monomorphism. It is called an extension of the structure group iff f is an identification map and φ is an epimorphism [G' is then a quotient group of G, or G is a group extension of G' by $\mathrm{Ker}(\varphi)$].

(b) Use will be made of the following result (Bott & Mather, 1967).

Classification theorem. There is a 1–1 correspondence between isomorphism classes of principal fibre bundles with structure group G over a topological space X and the Čech cohomology group $\check{H}^1(X, \mathbf{G})$.

Roughly, the proof goes as follows. Let ξ be a principal fibre bundle over X with structure group G. Each point of X has a neighbourhood V along with a local section s_V of the bundle projection $\pi: E \to X$ over V

$$\pi \circ s_V = i_V : V \subsetneq X$$

Moreover, all such opens form a base for the topology of X. If $v' \in \mathrm{Cov}(X)$, there is a $v < v'$ and for each $V \in v$ a local section s_V of π. Suppose that v is such a cover. One defines a multiplicative Čech cocycle g_v of v in $\mathbf{G}$ as follows. If $x \in V \cap V'$ for $V', V \in v$, since s_V and $s_{V'}$ are sections and G acts transitively on the fibre over x, there is a $g^s_{V,V'}(x) \in G$ with $s_V(x) = s_{V'}(x)\,.\,g^s_{V'V}(x)$. If $x \in V \cap V' \cap V''$ for $V, V', V'' \in v$, we must have

$$g^s_{VV''}(x) = g^s_{VV'}(x)\,.\,g^s_{V'V''}(x)$$

which is the multiplicative cocycle condition. If s' is another system of local sections on v, then for each $V \in v$ and $x \in V$ there is a $g^{s'}_{sV}(x) \in G$ such that

$$g^{s'}_{VV'}(x) = g^{s'}_{sV}(x)\,.\,g^s_{VV'}(x)\, g^{s'}_{sV'}(x)^{-1}$$

Thus the cocycles $g_v{}^s$ and $g_v{}^{s'}$ of v in $\mathbf{G}$ are cohomologous, and the cohomology class $g_v = |g_v{}^s| \in H^1(v, \mathbf{G})$ depends only on v. Let π^v: $H^1(v, \mathbf{G}) \to \check{H}^1(X, \mathbf{G})$ be the projections of the direct limit. Then for any $v \in \mathrm{Cov}(X)$, $\pi^v(g_v) \equiv g_\xi \in \check{H}^1(X, \mathbf{G})$ depends only on ξ. Moreover, if ξ' is a principal fibre bundle with structure group G over X and if $\xi \cong \xi'$, $g_\xi = g_{\xi'}$. We have thus obtained a function from isomorphism classes of principal fibre bundles with structure group G over X to $\check{H}^1(X, G)$. That the function is a bijection is shown by constructing an inverse function.

2.3. *The Einstein Bundle of Bases and its Reductions*

(a) We shall be interested in the reduction of the Einstein bundle (Bräuer, 1971) of bases (Hicks, 1964) $\xi_E(X)$ of a space-time X. The Einstein bundle $\xi_E(X)$ has total space $\mathscr{B}_E(X)$ of all bases b of all the tangent (Minkowski) spaces $T(X,x)$ over the points x of X. It has structure group $GL(4,\mathbb{R})$ (in this context the Einstein group E). The local sections of the bundle projection are defined in terms of the atlases 'A' of X. If A is an atlas and $V = \text{dom}(y)$ is a coordinate domain in A, there is a local section $S_V{}^A: x \mapsto (\partial/\partial y)x$ of the projection. The overlap cocycles are defined on $V \cap V' \equiv \text{dom}(y) \cap \text{dom}(y')$ by $g^A_{VV'}: x \mapsto (\partial y'/\partial y)x$.

(b) The Einstein bundle can always be reduced to L the (full) Lorentz group $L \subset E$

$$O(1,3,\mathbb{R}) \subset GL(4,\mathbb{R})$$

(Markus, 1955). To see this, let $g \in GL(4,\mathbb{R})$ be the diagonal matrix with entries $+1$, -1, -1, -1. Define $\mathscr{B}_L(X) \subset \mathscr{B}_E(X)$ as the set of bases b of $\mathscr{B}_E(X)$ such that

$$\mathscr{L}_{\pi(b)} = g_{ij} b_i{}^* \otimes b_j{}^*$$

where b^* is the dual basis to b of the contangent space $T^*(X, \pi(b))$. If b, $b' \in \mathscr{B}_L(X)$ and $\pi(b) = \pi(b')$ there is $\Lambda \in E$ with $b' = b.\Lambda$. But then $g.\Lambda.g^{-1} = \Lambda^{-1t}$ which implies that $\Lambda \in O(1,3,\mathbb{R})$. $\mathscr{B}_L(X)$ is the total space of a principal fibre bundle $\xi_L(X)$ over X with structure group L, the Lorentz bundle. We aim to discuss the further reducibility of $\xi_L(X)$ below and link it with the orientability properties discussed in Part 1.

2.4. *Group Extensions and the Structure of L*

(a) A group G is called a group-extension of a group Q by a group K iff all three lie on a short exact sequence of groups:

$$K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} Q$$

Through the inner automorphisms of G, there is a canonical homomorphism $w \in \text{Hom}(Q, \text{Out}(K))$, where $\text{Out}(K)$ is the group of outer automorphisms of K: $\text{Aut}(K)/\text{Int}(K)$. [$\text{Aut}(K)$ is the group of all automorphisms of K and $\text{Int}(K) \lhd \text{Aut}(K)$ is the group of inner automorphisms.] In fact, $\text{Aut}(K)$ is a group extension of $\text{Out}(K)$ by $\text{Int}(K)$. A group extension is said to 'split' on the right iff there is a homomorphism $\gamma: Q \to G$ which is a section of β over Q. If the group extension $\text{Aut}(K)$ of $\text{Out}(K)$ by $\text{Int}(K)$ splits on the right, there is a lift w' of w in any group extension of Q by K to $\text{Aut}(K)$. If, further, the original group extension splits on the right, it is called a semi-direct product of Q by K and is written $KX_{W'}Q$. The latter is completely determined (up to equivalence) by K, Q and $w' \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$. When w' is the trivial action, Q is an invariant subgroup of G, and G is called the direct product of Q by K written $K \times Q$. $K \times Q$ is also a group extension of K by Q. Conversely, if we are given an action $w \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$ of Q on K, we may define a semi-direct product of Q by K on the *set* $K \times Q$ by writing

$$(k_1, q_1).(k_2, q_2) = (k_1 . w(q_1)(k_2), q_1 . q_2)$$

A group extension of Q by K is called central iff the canonical homomorphism $w \in \mathrm{Hom}(Q, \mathrm{Out}(K))$ is trivial. (In this case, if K is abelian it is a central subgroup of the group extension.)

Alternatively, a group extension of Q by K may be specified by a pair (G, s), where s is an epimorphism of G onto Q and $\mathrm{Ker}(s) \lhd G$ is isomorphic to K. We use this notation below.

(b) the Lorentz group L has the following structures as a group extension (Michel, 1965) corresponding to the invariant subgroups L_+, $L\!\uparrow$ and $L_+\!\uparrow = L_+ \cap L\!\uparrow$ of L; respectively the proper, the orthochronous and the proper orthochronous Lorentz groups. These latter are defined by $L\!\uparrow = \mathrm{Ker}(t)$ for $t: L \to \mathbb{Z}_2$; $t: \mapsto \Lambda_{00}/|\Lambda_{00}|$ and $L_+ = \mathrm{Ker}(d)$ for $d: L \to \mathbb{Z}_2$; $d: \Lambda \mapsto \det(\Lambda)$. The extension structures are:

(i) $(L, t) = L\!\uparrow X_W \mathbb{Z}_2(T)$. Here, $\mathbb{Z}_2(T)$ is the two-element cyclic group generated by the 'time-inversion' $T = -g$ [g is defined in Section 2.3(a)]. $W \in \mathrm{Hom}(\mathbb{Z}_2(T), \mathrm{Aut}(L\!\uparrow))$ is defined by $W(T): \Lambda \mapsto T.\Lambda.T^{-1} = \Lambda^{-1t}$. If I is the space-time inversion $-\mathbb{1}$, one may re-express the above structure by $L\!\uparrow X \mathbb{Z}_2(I)$. These two extension structures are equivalent, even though the former does not 'look' central. In fact, $W(T) = \mathrm{In}(g)$, the inner automorphism of $L\!\uparrow$ induced by $g \in L\!\uparrow$.

(ii) $(L, d) = L_+ X_{W'} \mathbb{Z}_2(g)$. $\mathbb{Z}_2(g)$ is the two-element cyclic group generated by g, being the homomorphic image in L of the section $-1 \mapsto g$ of d: $L \to \mathbb{Z}_2$. The action $W' \in \mathrm{Hom}(\mathbb{Z}_2(g), \mathrm{Aut}(L_+))$ is just $W'(g)$: $\Lambda \mapsto g.\Lambda.g^{-1} = \Lambda^{-1t}$. The extension is again central because $W'(g) - \mathrm{In}(T)$ for $T \in L_+$.

(iii) $(L, f) = L_+\!\uparrow X_{W''}(\mathbb{Z}_2(g) \times \mathbb{Z}_2(T))$. The homomorphism f from L onto $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $f: \Lambda \mapsto (d(\Lambda), t(\Lambda))$ and by definition, $\mathrm{Ker}(f) = \mathrm{Ker}(d) \cap \mathrm{Ker}(t) = L_+\!\uparrow$. Alternatively, we may express the extension as $(L_+ X_p \mathbb{Z}_2(g)) \times \mathbb{Z}_2(I)$; $\mathbb{Z}_2(I)$ acting trivially on $L_+\!\uparrow$.

(c) As well as utilising the above structures of L, we shall be making use of an extension of $L_+\!\uparrow$ expressed by the short exact sequence of groups:

$$\mathbb{Z}_2 \underset{\text{inc.}}{\rightarrowtail} SL(2, \mathbb{C}) \underset{\pi}{\twoheadrightarrow} L_+\!\uparrow$$

π is the universal covering projection of $SL(2, \mathbb{C})$ onto $L_+\!\uparrow$. $SL(2, \mathbb{C})$ and $L_+\!\uparrow$ have topologies $\mathbb{R}^3 \times S^3$ and $\mathbb{R}^3 \times \mathbb{R}P^3$ respectively.

Spin(n) (Porteous, 1969) is the two-fold covering group of the orthogonal group $SO(n, \mathbb{R})$ and is a group extension of $SO(n, \mathbb{R})$ by $\mathbb{Z}_2$. We note that Spin$(3) = SU(2)$ and that there is a homomorphism of group extensions:

$$\begin{array}{ccccc} \mathbb{Z}_2 & \xrightarrow[\text{inc.}]{} & SL(2,\mathbb{C}) & \xrightarrow{\pi} & L_+\uparrow \\ \| & & \uparrow \text{inc.} & & \uparrow \text{inc.} \\ \mathbb{Z}_2 & \xrightarrow[\text{inc.}]{} & \text{Spin}(3) & \xrightarrow[\pi']{} & SO(3) \end{array}$$

Spin (3) and $SO(3, \mathbb{R})$ are respectively the maximal compact subgroups of $SL(2, \mathbb{C})$ and of $L_+\uparrow$.

2.5. *Reducibility of $\xi_L(X)$ to $L\uparrow$*

(a) Corresponding to the structure of L as a group extension of $\mathbb{Z}_2(T)$ by $L\uparrow$ the orthochronous Lorentz group, there is a short-exact sequence:

$$\mathbf{L}\uparrow \xrightarrow{\alpha} \mathbf{L} \xrightarrow{\beta} \mathbb{Z}_2$$

of constant presheaves of groups over any space-time X. Let γ be the associated splitting homomorphism on the right. As we have seen, the above exact sequence of presheaves induces a long-exact 'Bockstein' coefficient sequence of Čech cohomology groups. A portion of this Bockstein sequence is:

$$\dashrightarrow H^1(X, \mathbf{L}\uparrow) \xrightarrow{\alpha_*} H^1(X, \mathbf{L}) \underset{\gamma_*}{\overset{\beta_*}{\rightleftarrows}} H^1(X, \mathbb{Z}_2) \dashrightarrow$$

The homomorphism β_* is an epimorphism because $\beta_* \circ \gamma_* = \mathbb{1}$. Therefore, the Čech group $\check{H}^1(X, L)$ is a (split) group extension of $\check{H}^1(X, \mathbb{Z}_2)$ by $\alpha_*(\check{H}^1(X, \mathbf{L}\uparrow))$. Any element of $\check{H}^1(X, \mathbf{L})$, including X_L the cohomology class representing the isomorphism class of the Lorentz bundle $\xi_L(X)$ of bases over X, can be written as a product of an element of $\check{H}^1(X, \mathbf{L}\uparrow)$ and an element of $\check{H}^1(X, \mathbb{Z}_2)$

$$X_L = \alpha_*(X_{L\uparrow}) \cdot \gamma_*(\psi_X)$$

The element $X_L\uparrow$ of $\check{H}^1(X, \mathbf{L}\uparrow)$ represents the isomorphism class of a principal fibre bundle over X with structure group $L\uparrow$ and $\psi_X \in \check{H}^1(X, \mathbb{Z}_2)$ the isomorphism class of a principal fibre bundle over X with structure group $\mathbb{Z}_2(T)$. Clearly, $\xi_L(X)$ reduces to $L\uparrow$ iff $\psi_X = 0$, since γ_* is a monomorphism. Therefore, the Čech cohomology class ψ_X also represents the obstruction to the reducability of $\xi_L(X)$ to $L\uparrow$. We show in the next section that the principal fibre bundle over X with structure group $\mathbb{Z}_2$ represented by ψ_X is isomorphic to the time-sense bundle $\tau(X)$ over X; therefore, $\xi_L(X)$ reduces to $L\uparrow$ iff X is time-orientable.

(b) Recall that the total space of the Lorentz bundle of bases of a space-time X is $\mathscr{B}_L(X)$,the totality of bases of the tangent Minkowski spaces over

X which diagonalise the Lorentzian tensor. Define an equivalence relation $(\approx)$ in $\mathscr{B}_L(X)$ by writing $b \approx b'$ iff $\pi(b) = \pi(b')$ and $b' = b.\Lambda$ for $\Lambda \in L$. The quotient space of $\mathscr{B}_L(X)$ under this equivalence relation is a principal fibre bundle over X with structure group $\mathbb{Z}_2$. The relation $(\approx)$ divides $\mathscr{B}_L(X)$ into two families, the bases with respectively 'positive' or 'negative' temporal orientations. Over each point of X there is a pair of equivalence classes of bases of the tangent Minkowski space $T(X,x_0)$: $\mu(b_0)$ and $\mu(b_0.T).b_0$ is any base of $T(X,x_0)$ which includes the unit time-like vector θ_x ($b_0.T$ then includes $-\theta_x$). The function $\mu(b_0) \mapsto \nu(b_0{}^0)$ is then an isomorphism of principal fibre bundles over X. That is, $\psi_X \in \check{H}^1(X,\mathbf{Z}_2) \cong H^1(X,\mathbb{Z}_2)$ (Spanier, 1966) represents the isomorphism class of $\tau(X)$ and $\xi_L(X)$ reduces to $L\!\uparrow$ iff X is time-orientable.

2.6. *Reducibility of $\xi_L(X)$ to L_+*

(a) Corresponding to the structure of L as a semi-direct product of $\mathbb{Z}_2(g)$ by L_+, there is a split-exact sequence of presheaves of groups over X:

$$\mathbf{L}_+ \overset{\alpha'}{\rightarrowtail} \mathbf{L} \underset{\gamma}{\overset{\beta'}{\twoheadrightarrow}} \mathbf{Z}_2(g)$$

As above, there is a long-exact Bockstein sequence of Čech cohomology, the relevant portion of which is:

$$\dashrightarrow \check{H}^1(X,\mathbf{L}_+) \xrightarrow{\alpha'_*} \check{H}^1(X,\mathbf{L}) \underset{\gamma'_*}{\overset{\beta'_*}{\rightleftarrows}} \check{H}^1(X,\mathbf{Z}_2(g)) \dashrightarrow$$

β'_* is an epimorphism because there is a section γ' of β' and $\check{H}^1(X,\mathbf{L})$ is expressible as a split-extension of $\check{H}^1(X,\mathbf{Z}_2)$ by $\alpha'_*(\check{H}^1(X, \mathbf{L}_+))$. Let $X_L \in \check{H}^1(X,\mathbf{L})$ still represent the isomorphism class of the Lorentz bundle $\xi_L(X)$. Then we may write $X_L = \alpha'_*(X_{L_+})\cdot\gamma'_*(\psi_X')$ where $X_{L_+} \in \check{H}^1(X,\mathbf{L}_+)$ and $\psi_X' \in \check{H}^1(X,\mathbf{Z}_2)$ represent principal fibre bundles with structure groups L_+ and $\mathbb{Z}_2(g)$ respectively. Clearly, $\xi_L(X)$ reduces to L_+ iff $\psi_X' = 0$. We shall show below that ψ_X' in fact represents the isomorphism class of the bundle of orientations of X, or that $\xi_L(X)$ reduces to L_+ iff X is an orientable 4-manifold.

(b) To see this, recall the definition of orientability of a manifold in geometrical terms. A manifold X is called orientable (X is now smooth) iff its tangent bundle is an orientable bundle. Equivalently, X is orientable iff its bundle of bases reduces to the subgroup $GL(n,\mathbb{R})_+ \lhd GL(n,\mathbb{R})$, where $GL(n,\mathbb{R})$ is the structure group of X and $GL(n,\mathbb{R})_+$ is the kernel of the homomorphism:

$$\theta: GL(n,\mathbb{R}) \twoheadrightarrow \mathbf{Z}_2$$
$$\theta: g \mapsto \operatorname{sign}(\det(g)).$$

Thus corresponding to the short exact sequence of presheaves of groups over X:

$$\mathbf{GL}(n,\mathbb{R})_+ \overset{\alpha}{\rightarrowtail} \mathbf{GL}(n,\mathbb{R}) \overset{\beta}{\twoheadrightarrow} \mathbf{Z}_2$$

there is a corresponding long exact Čech cohomology sequence and a principal bundle over X with structure group $\mathbb{Z}_2$ (called the orientation bundle of X) such that X is an orientable manifold iff the orientation bundle is trivial. Specialising to $n=4$, consider the following diagram of group homomorphisms:

$$\begin{array}{ccccc} GL(4,\mathbb{R})_+ & \xrightarrow[\text{inc.}]{\alpha} & GL(4,\mathbb{R}) & \xrightarrow{\beta} & \mathbb{Z}_2 \\ \uparrow \text{inc.}\, j & & \uparrow \text{inc.}\, j' & & \| \\ O(1,3;\mathbb{R})_+ & \xrightarrow[\text{inc.}]{\alpha'} & O(1,3;\mathbb{R}) & \xrightarrow[\beta']{} & \mathbb{Z}_2 \end{array}$$

This induces a Čech cohomology ladder, a portion of which is

$$\begin{array}{ccccccc} \dashrightarrow & \check{H}^1(X,\mathbf{GL(4,R)}_+) & \xrightarrow{\alpha_*} & \check{H}^1(X,\mathbf{GL(4,R)}) & \xrightarrow{\beta_*} & \check{H}^1(X,\mathbf{Z}_2) & \dashrightarrow \\ & \uparrow j_* & & \uparrow j'_* & & \| & \\ \dashrightarrow & \check{H}^1(X,\mathbf{O(1,3;R)}_+) & \xrightarrow[\alpha'_*]{} & \check{H}^1(X,\mathbf{O(1,3;R)}) & \underset{\beta'_*}{\overset{\gamma'_*}{\rightleftarrows}} & \check{H}^1(X,\mathbf{Z}_2) & \dashrightarrow \end{array}$$

Consider the Čech cohomology class $\mathbf{j'}_*(X_L)\in\check{H}^1(X)$, $\mathbf{GL(4)}$). The class $\boldsymbol{\beta}_*(j'_*(X_L))$ represents the isomorphism class of the orientation bundle of X if X is a space-time. But $\boldsymbol{\beta}_*\circ j'_*(X_L)=\beta'_*(X_L)=\psi'_X$. Therefore X reduces to L_+: $\psi_X=0$, iff X is an orientable 4-manifold.

2.7. *Reducibility of $\xi_L(X)$ to $L_+\uparrow$*

Here, one makes use of the structure of L as a split extension of $\mathbb{Z}_2(g)\times\mathbb{Z}_2(T)$ by $L_+\uparrow$. there is a corresponding diagram of presheaves of groups over X and a long exact 'Bockstein' sequence of Čech cohomology groups, the relevant portion of which is

$$\dashrightarrow \check{H}^1(X,\mathbf{L}_+\uparrow) \xrightarrow{\alpha''_*} \check{H}^1(X,\mathbf{L}) \underset{\gamma''_*}{\overset{\beta''_*}{\rightleftarrows}} \check{H}^1(X,\mathbf{Z}_2\times\mathbf{Z}_2) \dashrightarrow$$

and $\check{H}^1(X,\mathbf{L})$ is a (split) extension of $\check{H}^1(X,\mathbf{Z}_2\times\mathbf{Z}_2)\simeq\check{H}^1(X,\mathbf{Z}_2)\oplus\check{H}^1(X,\mathbf{Z}_2)$ by $\alpha''_*(\check{H}^1(X,\mathbf{L}_+\uparrow))$. The class of the Lorentz bundle $X_L\in\check{H}^1(X,\mathbf{L})$ can therefore be written $X_L=\alpha''_*(X_{L_+}\uparrow)\gamma''_{1*}(\psi_X)\cdot\gamma''_{2*}(\psi'_X)$ where γ''_1, γ''_2 are composites of monomorphisms and hence monomorphisms. Thus $\xi_L(X)$ reduces to $L_+\uparrow$ iff ψ_X and ψ'_X vanish iff X is both an orientable manifold and a time-orientable space-time.

There is the following connection between the overall orientability of a space-time manifold and the properties of time and space-orientability. The orientation bundle of a space-time is a fibre-product of the time-orientation bundle and the space-orientation bundle. This means that any two of the three possible orientability properties imply the third. For example, if a space-time is time-orientable and is an orientable 4-manifold, it is a space-orientable space-time. Also, if X is time-orientable and space-orientable, it is also an orientable 4-manifold and its structure group reduces to $L_+\uparrow$.

2.8. *Spinor Structures*

A space and time orientable space-time X is said to carry a spinor structure iff its Lorentz bundle $\xi L_+\uparrow(X)$ can be extended to $SL(2,\mathbb{C})$ in such a way that the covering map from the bundle of 'spinor bases' (Penrose, 1968) of X to the bundle of bases restricts on the fibres $SL(2,\mathbb{C})$ to be the universal cover onto $L_+\uparrow$. We first briefly discuss a closely allied concept 'spin structures'.

(a) A principal fibre bundle ξ with structure group $SO(n,\mathbb{R})$ over a space X is said to carry a 'spin-structures' (Borel & Hirzebruch, 1959; Milnor, 1963) iff ξ can be extended to Spin(n) in such a way that the projection restricted to a fibre is the covering map of Spin(n) onto $SO(n,\mathbb{R})$. Corresponding to the exact sequence:

$$\mathbf{Z}_2 \rightarrowtail \mathbf{Spin}(n) \xrightarrow{\pi} \mathbf{SO}(n,\mathbb{R})$$

of presheaves over X is the usual coefficient sequence of Čech cohomology, a portion of which is:

$$\to \check{H}^1(X,\mathbf{Spin}(n)) \xrightarrow{\pi_*} \check{H}^1(X,\mathbf{SO}(n,\mathbb{R})) \xrightarrow{\delta_*} \check{H}^2(X,\mathbb{Z}_2)$$

If $\hat{\xi} \in \check{H}^1(X,\mathbf{SO}(n,\mathbb{R}))$ is the representative of the isomorphism class of ξ, ξ has a spin structure $\hat{\xi} \in \mathrm{Im}(\pi_*)$ iff $\hat{\xi} \in \mathrm{Ker}(\delta_*)$ by exactness, or iff $\delta_*(\hat{\xi}) = 0$. That is, ξ carries a spin structure iff $\delta_*(\hat{\xi}) \in \check{H}^2(X,\mathbb{Z}_2)$ vanishes. In the case that X is a compact manifold, it can be shown that $\delta^*(\hat{\xi}) = w_2(X)$, the second Stiefel–Whitney class of X. Hence, a principal $SO(3,\mathbb{R})$ bundle over a compact space carries a spin-structure iff $w_2(X) = 0$.

(b) A space and time orientable space-time X carries a spinor-structure iff its Lorentz bundle of bases $\xi_{L_+\uparrow}(X)$ extends to $SL(2,\mathbb{C})$ in such a way that the restriction of the covering map to any $SL(2,\mathbb{C})$ fibre is the universal covering map $SL(2,\mathbb{C}) \to L_+\uparrow$. Recall that there is a morphism of group extensions

$$\begin{array}{ccccc}
\mathbb{Z}_2 & \xrightarrow{\text{inc.}} & SL(2,\mathbb{C}) & \xrightarrow{\pi} & L_+\uparrow \\
\| & & \uparrow{\scriptstyle \tau=\text{inc.}} & & \uparrow{\scriptstyle \tau'=\text{inc.}} \\
\mathbb{Z}_2 & \xrightarrow[\text{inc.}]{} & \mathrm{Spin}(3) & \xrightarrow[\pi']{} & \mathrm{SO}(3,\mathbb{R})
\end{array}$$

where π and π' are the universal covering projections and Spin(3) and $SO(3)$ are the maximal compact subgroups of respectively $SL(2,\mathbb{C})$ and $L_+\uparrow$ which are connected Lie groups. There is an induced diagram of presheaves of groups over X and a Čech cohomology ladder

$$\begin{array}{ccccccccc} \cdots\longrightarrow & \check{H}^1(X,\mathrm{SL}(2,\mathbb{C})) & \xrightarrow{\pi_*} & \check{H}^1(X,L_+\!\uparrow) & \xrightarrow{\delta_*} & \check{H}^1(X,\mathbb{Z}_2) & \longrightarrow\cdots \\ & \uparrow{\scriptstyle \tau_*} & & \uparrow{\scriptstyle \tau_*'} & & \| & \\ \cdots\longrightarrow & \check{H}^1(X,\mathrm{Spin}(3)) & \xrightarrow[\pi_*']{} & \check{H}^1(X,\mathrm{SO}(3)) & \xrightarrow[\delta_*']{} & \check{H}^1(X,\mathbb{Z}_2) & \longrightarrow\cdots \end{array}$$

The vertical homomorphisms are isomorphisms because Spin(3) and $SO(3,\mathbb{R})$ are the maximal compact subgroups of the connected Lie groups $SL(2,\mathbb{C})$ and $L_+\!\uparrow$ (Bott & Mather, 1967). [Incidentally, $\xi_{L_+\uparrow}(X)$ can therefore always be reduced to $SO(3,\mathbb{R})$.] If X_L as usual represents the isomorphism class of the Lorentz bundle in $\check{H}^1(X,L_+\!\uparrow)$, then X carries a spinor structure iff the reduction of $\xi_{L_+\uparrow}(X)$ to $SO(3,\mathbb{R})$ carries a *spin* structure. Therefore, if X is a compact space-time it carries a spinor structure iff its second Stiefel–Whitney class vanishes.

(c) Geroch (1968) has shown that if a non-compact space-time carries a spinor structure then it is the trivial one. To do so he constructed a global section of the 'spinor bundle' of bases covering the Lorentz bundle. Therefore, projecting this section on the Lorentz bundle to yield a global section of the latter, we obtain a parallelisation of X. Thus a space-time which admits a spinor structure must be parallelisable. Conversely, a parallelisable space and time orientable space-time admits a spinor structure, because if X is parallelisable, $\mathscr{B}_{L+\uparrow}(X) \cong X \times L_{+\uparrow}$ and the obvious projection:

$$X \times SL(2,\mathbb{C}) \to X \times L_{+\uparrow}$$

is a spinor structure of X.

This concludes Part 2 and our analysis of space-time orientability. Part 3 is given over to a discussion of a different topic.

Part 3. Coarse Causal Topologies on Space-Times

The purpose of this part of the present paper is to describe an extension of some results published earlier (Whiston, 1972) in the form of letter. In that letter; the retopologisation of Minkowski space with a topology induced by Zeeman's causal order (Zeeman, 1964) was discussed. The generalisation described here is to retopologise time-orientable space-times with analogous topologies generated by their causal relations.

Let us agree to call a time-orientable space-time X 'future complete' iff its unit-timelike vector field is right complete or 'past-complete' iff the vector field is left complete. For example; the open top half of a two-dimensional Minkowski space whose Lorentz tensor is generated by $(\partial/\partial y)$ is future, but not past complete. A curve c in X is called time-like and oriented towards the future iff its canonical lift to $T(X)$, $c = T(c) \circ (d/dt)$ lies in the open 'sub-bundle' of future-time cones.

A causal relation may be defined in X as follows (Geroch, 1969; Penrose, 1968). If $x, x' \in X$ one writes $x < x'$ iff x and x' lie on a smooth, time-like curve oriented towards the future and $x, x' = c(s), c(s')$ for $s < s'$ in $\mathrm{dom}(c)$. The relation $(<)$ is transitive; for if $x < x'$ and $x' < x''$ we note that x and x'' lie on a continuous curve; smooth except perhaps at x' where tangent vector expect perhaps at x' is time-like and oriented towards the future with $x, x'' = c''(s), c''(s'')$ for $s < s''$ in $\mathrm{dom}(c'')$. To obtain the required curve one rounds off c'' at x'. This implies $x < x''$.

The topology generated by $(<)$: $\tau(<)$ has for basis all the 'pasts' $\langle x] \equiv \{y \in X \vdash y \leqslant x\}$ of the points of X. That the latter family forms a basis, that the singleton collections $\{\langle x]\}$ form a local base at any $x \in X$, that the 'futures' $[x\rangle \equiv \{y \in X \vdash x \leqslant y\}$ are $\tau(<)$-closed and that for any $x \in X$, $Cl(< x]) = \{y \in X \vdash < y] \vdash < y] \cap < x] \neq \varnothing\}$, are easy consequences of the transitivity of $(<)$. X retopologised with $\tau(<)$ will be denoted by $X(<)$. $\tau(<)$ is rather a weird and, unhappily, not too useful, topology on a space-time. Compare it with, say, the Alexandroff topology (Penrose, 1968). Whilst $\tau(<)$ is not comparable to the manifold topology, the Alexandroff topology is always coarser and sometimes even coincident with the manifold topology. However, $\tau(<)$ does have some interest in that it allows one to couch properties of the causal order in topological terms.

Proposition 1. $X(<)$ is T_0 iff X admits no closed future-oriented time-like curves.

Proof. If $X(<)$ is not T_0 there exist points $x, x' \in X$, $x \neq x'$ and $x' \in V$, $x \in V'\ \forall\ V \in N(x)$, $V' \in N(x')$. In particular, $x \in\ < x']$ and $x' \in\ < x]$, which means that $x \leqslant x' \leqslant x$. Because $x \neq x'$, $x < x' < x \Rightarrow x < x$ or x lies on a closed future-oriented time-like curve. Conversely, let c be a closed, future-oriented time-like curve in X and x, x' be distinct points on c. Then $x < x'$ and $x' < x$ implying $x \in\ < x']$ and $x' \in\ < x]$ or $x \in V'$, $x' \in V\ \forall\ V, V' \in N(x), N(x')$. $X(<)$ cannot therefore be T_0.

By the above, if X is a compact space-time, $X(<)$ is not T_0, T_1 or T_2. In fact, $X(<)$ can never be T_1 (or T_2). In most cases $X(<)$ is not T_3 or T_4; for if there are $x, x' \in X$ such that $x \in X \backslash [x'\rangle$ and $< x] \cap < x'] \neq \varnothing$, any two neighbourhoods. $V \in N(x)$, $V' \in N([x'\rangle)$ meet. This is because the open set

$$\bigcup_{x'' \geqslant x'} \{< x'']\}$$

is a basis for the neighbourhoods of $[x'\rangle$. By hypothesis

$$< x] \cap \Big(\bigcup_{x'' \geqslant x'} \{< x'']\}\Big) \neq \varnothing$$

In some cases, $X(<)$ is hyperconnected (Steen & Seebach, 1970), that is $X(<)$ has no disjoint open sets. Clearly, if $X(<)$ is hyperconnected it is connected and cannot be T_3. Moreover, hyperconnected spaces are compact iff they are locally compact iff they are paracompact. For example, if X has a null past infinity, $X(<)$ is hyperconnected because for any points x, $x' \in X$, $<x] \cap <x'] \neq \varnothing$; whilst if X has a space-like past infinity (Penrose, 1964), there are points 'near the boundary' whose pasts do not intersect.

Proposition 2. If X is future complete, $X(<)$ is non-compact. Proof: Let c_x^V be the maximal integral curve of the right complete vecto field V from $x \in X$, and let $\{x_n\}n \in \mathbb{N}$ be the integer points along it. $\{[x_n\rangle\}n \in \mathbb{N}$ is a family of closed sets of $X(<)$ with the finite intersection property: $\cap\, n \in K\{[x_n>\} \neq \varnothing$ if K is a finite set of integers. But $\cap\, n \in \mathbb{N}\{[x_n>\} = \varnothing$. For if not, $[x>$ would be bounded above, which is not true.

Lastly, it follows trivially from the definition of homoeomorphism and the fact that $\{< x]\}$ is a local basis at any point $x \in X$, that the group of auto-homoeomorphisms of $X(<)$ is the group of all permutations of X which bipreserve the causal order

$$x < x' \Rightarrow f(x) < f(x') \qquad \text{and} \qquad f^{-1}(x) < f^{-1}(x')$$

For Minkowski space, Zeeman's theorem (Zeeman, 1964) states that $\mathrm{aut}(X(<)) = D\!\uparrow$, the orthochronous Poincaré-dilatation group.

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References

Aharanov, Y. and Susskind, L. (1967). *Physical Review*, **159**, 1237.
Borel, A. and Hirzebruch, F. (1959). *American Journal of Mathematics*, **81**, 315.
Bott, R. and Mather, J. (1968). *Proceedings of the Battelle Rencontres in Mathematics and Physics, 1967*, ed. De-Witt, B. and Wheeler, J. A. W. A. Benjamin.
Bräuer, H. J. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 4, p. 243.
Brickell, F. and Clarke, R. S. (1970). *Differentiable Manifolds*. Van-Nostrand Reinhold.
Eilenberg, S. (1963). Algebraic Topology. *Lectures in Modern Mathematics*, ed. Saaty, T. L. John Wiley and Sons.
Eilenberg, S. and Maclane, S. (1947). *Annals. of Mathematics*, **48**(2).
Geroch, R. (1968). *Journal of Mathematical Physics*, **9**(11), 1739.
Geroch, R. (1969). Space-Time Structure from a Global Viewpoint. *Proceedings of the Enrico-Fermi Summer School*. Academic Press.
Goldstein, R. Z. and Lininger, L. (1970). *Proceedings of the 1969 Georgia Topology Institute*, eds. Cantrell, and Edwards. Markham.
Greenberg, M. J. (1967). *Lectures on Algebraic Topology*. W. A. Benjamin.
Hicks, N. (1964). *Notes on Differential Geometry*. Van-Nostrand.
Lichnerowicz, A. (1968). Topics on Space-Times. *Proceedings of the Battelle Rencontres in Mathematics and Physics, 1967*, ed. De-Witt, B. and Wheeler, J. A. W. A. Benjamin.
Markus, L. (1955). *Annals of Mathematics*, **63**(2), 411.
Maunder, C. R. F. (1970). *Algebraic Topology*. Van-Nostrand Reinhold.

Michel, L. (1965). Relations Entre Symétries Intérnes et Invariance Relativiste. *Proceedings of the Cargese summer school, 1965*, ed. Lurçat, F. Gordon and Breach.
Milnor, J. (1963). *L'Ensiegnement Mathématique*, 9, 198.
Penrose, R. (1964). A Conformal Treatment of Infinity. *Relativity, Groups and Topology*, ed. De-Witt, B. Gordon and Breach.
Penrose, R. (1968). The Structure of Space-Time. *Proceedings of the Battelle Rencontres in Mathematics and Physics, 1967*, ed. De-Witt, B. and Wheeler, J. A. W. A. Benjamin.
Porteous, I. R. (1969). *Topological Geometry*. Van-Nostrand Reinhold.
Scott, W. (1964). *Group Theory*. Prentice-Hall.
Spanier, E. (1966). *Algebraic Topology*. McGraw-Hill, New York.
Steen, L. A. and Seebach, J. A. Jnr. (1970). *Counterexamples in Topology*. Holt, Rinehart and Winston, New York.
Whiston, G. S. (1972). *International Journal of Theoretical Physics*, Vol. 6, No. 1, pp. 75–76.
Zeeman, E. C. (1964). *Journal of Mathematical Physics*, 5(4), 460.